Zeroth-Order Feedback Strategies for Medium-Range Interception in a Horizontal Plane

J. Shinar*
Technion—Israel Institute of Technology, Haifa, Israel

A zeroth-order feedback solution of a variable-speed interception game between two aircraft in the horizontal plane, obtained by using the method of forced singular perturbations, is compared with the exact open-loop solution. The comparison indicates that the forced singular-perturbation approximation, based on multiple time-scale separation of the state variables, is satisfactory. Nevertheless, in cases where the interception terminates at speeds much lower than the maximum speed predicted by the reduced-order game, a rough estimate of these final speeds is needed to improve the accuracy. Using such estimates, a payoff error of 1% or less is demonstrated in all examples with sufficiently large initial distances of separation (greater than 6-8 times the best initial turning radius of the pursuer). The results validate the zeroth-order forced singular-perturbation approximation for medium-range air combat analysis. This explicit feedback strategy is a very attractive candidate for airborne implementation.

Nomenclature

a = speed of sound
C_D = total drag coefficient
C_{D_0} = zero lift drag coefficient
C_L = lift coefficient
D = total aerodynamic drag force
D_0 = zero lift drag force
D_i = induced-drag force in level flight
d = distance of capture
e = evader strategy
g = acceleration of gravity
h = altitude
J = cost function of the game
K = induced-drag parameter
L = aerodynamic lift force
M = Mach number
n = aerodynamic load factor defined by Eq. (12)
P = atmospheric pressure
p = pursuer strategy
q = dynamic pressure defined by Eq. (15)
R = distance of separation (range)
r = best turning radius at a given velocity
S = aircraft wing surface
T = thrust force
t = time
V = velocity
W = aircraft weight
X, Y = Cartesian coordinates
β = angular error defined in Fig. 2
ϵ = singular-perturbation parameter
ϵ_g = geometric perturbation parameter defined in Eq. (42)
ξ = throttle control parameter
χ = aircraft flying direction (azimuth)
ψ = line-of-sight angle, defined in Fig. 1

Subscripts

E = evader
f = final value
min = minimal value
max = maximal value
P = pursuer
s = sustained
0 = initial value

Superscripts

c = composite zeroth-order solution
r = reduced-order solution
$(\ )^*$ = optimal value
$(\hat{\ })$ = nondimensional distance

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*Professor, Department of Aeronautical Engineering. Associate Fellow AIAA.

I. Introduction

THERE is little doubt that the differential game approach is the most realistic mathematical formulation for air-to-air combat problems. However, it was pointed out in a recent review paper[1] that "... this approach is so complex that results to date have been disappointing." Some hope for a relative breakthrough in this area has arisen with the application of singular-perturbation techniques for nonlinear zero-sum pursuit-evasion games.[2-7] This method has created the potential to generate approximate feedback strategies for an important class of air combat problems, namely, medium-range air-to-air interception. (In general, interception ranges are characterized by the firing envelope of the respective weapon systems. In this context "medium range" is a loose term covering a domain of weapon delivery between 4 and 20 km, excluding both long-range missiles, such as the AIM-54 Phoenix, and guns or other typical short-range weapons.)

The approximate solutions obtained by the method of forced singular perturbations (FSP) seem to be very attractive candidates for airborne implementation. The practical usefulness of such an explicit feedback approximation has to be evaluated not only by its feasibility, but more importantly by its accuracy. For accuracy assessment of any approximation, the knowledge of the exact solution is required. In the last decade, several numerical algorithms were developed to solve the multidimensional, nonlinear, two-point boundary-value problem formulated by the set of necessary conditions of game optimality.[8-14] Even if it is virtually impossible to prove that the converged numerical solution of the necessary conditions is indeed optimal, that is, that it satisfies the sufficiency conditions for saddle-point optimality,[15] the only available way to assess the accuracy of an approximation seems to be by comparing it with an outcome of some iterative computing algorithm. Unfortunately, most of the computer programs[8-12] developed for such purposes are not in operational status. The

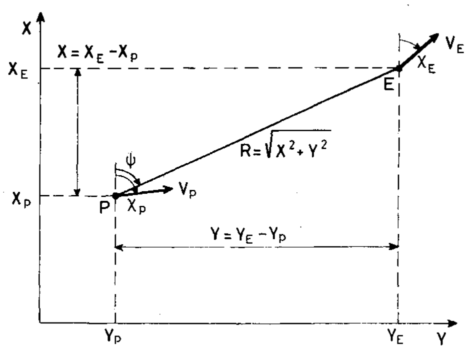

Fig. 1 Geometry of horizontal interception.

presently active computing algorithms[13,14] were also unable to provide the required comparison. Communication with the research personnel involved revealed that the algorithms currently are limited to the solution of fixed-time problems. Moreover, there is a major problem of convergence for the relatively long duration of a medium-range interception.[16]

Recently, a new opportunity has appeared for validating the zeroth-order feedback approximation obtained by using singular-perturbation methods. An open-loop solution of the variable-speed aircraft vs aircraft pursuit-evasion games in a horizontal plane was published in several recent papers.[17-19] The open-loop solution method, based on backward integration, allows a relatively detailed comparison with a feedback approximation of the same problem.[3] It has to be noted that the FSP approximation was also developed for the more realistic three-dimensional version of the above-mentioned pursuit-evasion game,[6,7] but its accuracy could not yet be tested because there is no optimal solution available for comparison.

The objective of the present paper is, therefore, to report the results of the comparison for a two-dimensional (horizontal) game, concentrating on the accuracy evaluation of the zeroth-order FSP approximation.

II. Formulation of the Air-to-Air Interception in the Horizontal Plane

A. Equations of Motion

The geometry of the relative position of two airplanes in a horizontal plane is depicted in Fig. 1. In an interception scenario, one airplane (having a superior weapons system) is the pursuer P, and the other is the evader E. The dynamics of the relative geometry can be expressed using the speed of sound a and the respective Mach numbers, either in polar coordinates,

$$\dot{R} = [-M_P\cos(\chi_P - \psi) + M_E\cos(\chi_E - \psi)]\,a; \quad R(t_0) = R_0 \tag{1}$$

$$\dot{\psi} = [-M_P\sin(\chi_P - \psi) + M_E\sin(\chi_E - \psi)]\,a/R; \quad \psi(t_0) = \psi_0 \tag{2}$$

or in a Cartesian coordinate system,

$$\dot{X} = (M_E\cos\chi_E - M_P\cos\chi_P)\,a; \qquad X(t_0) = X_0 \tag{3}$$

$$\dot{Y} = (M_E\sin\chi_E - M_P\sin\chi_P)\,a; \qquad Y(t_0) = Y_0 \tag{4}$$

by defining

$$X \triangleq X_E - X_P \tag{5}$$

$$Y \triangleq Y_E - Y_P \tag{6}$$

Aircraft dynamics for trajectory computations can be adequately represented by a point-mass approximation. In most studies[3-6,17-19] constant weight and thrust aligned with the velocity vector are assumed also. This set of assumptions leads to

$$\dot{M}_P = \frac{g}{a}\left(\frac{T}{W} - \frac{D}{W}\right)_P = \frac{g}{a}(\bar{T}_P - \bar{D}_P); \qquad M_P(t_0) = M_{P_0} \tag{7}$$

$$\dot{M}_E = \frac{g}{a}\left(\frac{T}{W} - \frac{D}{W}\right)_E = \frac{g}{a}(\bar{T}_E - \bar{D}_E); \qquad M_E(t_0) = M_{E_0} \tag{8}$$

$$\dot{\chi}_P = \frac{g}{aM_P}\,tg\mu_P = \frac{g}{aM_P}\left[\left(\frac{L}{W}\right)_P^2 - 1\right]^{\frac{1}{2}}; \qquad \chi_P(t_0) = \chi_{P_0} \tag{9}$$

$$\dot{\chi}_E = \frac{g}{aM_E}\,tg\mu_E = \frac{g}{aM_E}\left[\left(\frac{L}{W}\right)_E^2 - 1\right]^{\frac{1}{2}}; \qquad \chi_E(t_0) = \chi_{E_0} \tag{10}$$

The normalized propulsive thrust and the aerodynamic lift and drag forces can be expressed by

$$\frac{T}{W} \triangleq \bar{T} = \xi\bar{T}_{\max}(h, M) \tag{11}$$

$$\frac{L}{W} \triangleq n = q(h, M)\,SC_L/W \tag{12}$$

$$\frac{D}{W} = \bar{D} = q(h, M)\,SC_D(C_L, M)/W \tag{13}$$

where ξ is the throttle parameter in the range of

$$0 \le \xi \le 1 \tag{14}$$

q is the dynamic pressure defined (using the static pressure P) by

$$q(h, M) = 0.7P(h)\,M^2 \tag{15}$$

and the relationship between drag and lift is approximated by a parabolic polar

$$C_D(C_L, M)_0 = C_D(M) + K(M)\,C_L^2 \tag{16}$$

The maximum admissible value of the lift coefficient is limited by aerodynamic phenomena (e.g., stall, buffeting, or instability):

$$C_L \le C_{L_{\max}}(M) \tag{17}$$

The total lift force is also limited because of structural constraints. This limitation is usually expressed via the aerodynamic load factor

$$n \le n_{\max} \tag{18}$$

By combining Eqs. (12), (13), (16), and (17), the total drag-to-weight ratio can be expressed by

$$\bar{D} = \bar{D}_0 + n^2\bar{D}_i \tag{19}$$

$\bar{D}_0$ being the zero lift drag and $\bar{D}_i$ the induced drag for level flight ($n = 1$) divided by aircraft weight

$$\bar{D}_0 = qSC_{D_0}(M)/W \tag{20}$$

$$\bar{D}_i = K(M)\,W/qS \tag{21}$$

The admissible region of Mach numbers in horizontal flight is determined by a lower bound, a consequence of Eq. (17) with $n=1$, while the maximum Mach number can be either "placard" limited (because of structural and aerothermodynamical considerations) or the outcome of the thrust drag equilibrium in straight flight.

$$M_{\min} \leq M \leq M_{\max} \tag{22}$$

$$M_{\min} = \left[W/0.7P(h)SC_{L_{\max}}\right]^{\frac{1}{2}} \tag{23}$$

$$M_{\max} = \min\left(M_{\text{limit}}, \tilde{M} \triangleq \max M_{[\bar{T}_{\max}=\bar{D}_0+\bar{D}_i]}\right) \tag{24}$$

B. Differential Game Formulation

For such a formulation, the following elements must be defined: game space, game dynamics, admissible control sets, role determination of the players, information structure, game termination, and cost function.

For the game dynamics described by Eqs. (1), (2), and (7-10), the six-dimensional game space $(R, \psi, M_P, M_E, \chi_P, \chi_E)$ is bounded by

$$0 \leq R \leq \infty$$
$$(M_{\min})_P \leq M_P \leq (M_{\max})_P$$
$$(M_{\min})_E \leq M_E \leq (M_{\max})_E \tag{25}$$

The angular state variables (ψ, χ_P, ψ_E) are unbounded, but periodic; i.e.,

$$\psi + 2\pi = \psi, \qquad \chi_P + 2\pi = \chi_P, \qquad \chi_E + 2\pi = \chi_E \tag{26}$$

The control set of the players is (ξ_P, n_P) and (ξ_E, n_E), respectively, subject to Eqs. (14), (17), and (18). The roles of the players (pursuer and evader) are determined at the outset by the superior weapons system of the interceptor airplane. This role determination remains unchanged for the whole duration of the game.

The game starts at some initial conditions $(R_0, \psi_0, M_{P_0}, M_{E_0}, \chi_{P_0}, \chi_{E_0})$ at a moment $(t=0)$ when both players become aware of each other. For the whole duration of the game, perfect (memoryless) state information is assumed. The game terminates when the pursuer succeeds at the first time to approach the evader within a distance equal to the firing range of its weapon system. This condition is expressed by

$$R(t_f) = d \tag{27}$$

$$\dot{R}(t_f) < 0 \tag{28}$$

The terminal values of the other state variables are free. Thus, Eq. (27) describes a five-dimensional terminal manifold, and Eq. (28) determines the "usable part" of this surface. The time of game termination (or, in other words, the "time of capture") is determined, based on Eqs. (27) and (28), as

$$t_f = \min t_{[R=d]} \tag{29}$$

The cost function of the game is this time-of-capture,

$$J = t_f \tag{30}$$

The objective of the pursuer is to minimize t_f; the evader wants (if capture cannot be avoided) to maximize t_f.

The solution of this perfect information zero-sum differential game, for each set of initial conditions, is a triplet: a pair of optimal strategies $p^*(\cdot)$, $e^*(\cdot)$, and the saddle-point value of the cost function J^*, satisfying the inequality

$$J(p, e^*) \geq J(p^*, e^*) \triangleq J^* \geq J(p^*, e) \tag{31}$$

This quantitative game (game of degree) has a meaning only if capture, defined by Eqs. (27) and (28), can be guaranteed. This condition is obviously satisfied, even for a "tail-chase" configuration, if

$$M_P(t_f) > M_E(t_f) \tag{32}$$

Analysis of a simplified constant-speed pursuit-evasion game[20] indicates that capture can be achieved from any initial condition by a faster pursuer even against an evader of unlimited maneuverability if the ratio of the "capture range" to the pursuer's turning radius r_P exceeds the value of $[2(1-\sin 1)]^{\frac{1}{2}}$; i.e.,

$$d/r_P > 0.563 \tag{33}$$

In interceptions where "all-aspect" guided missiles are used, this inequality is easily satisfied. Thus, it can be concluded that the medium-range interception engagement described in the Introduction is a suitable example for the above-formulated differential game of degree. Such encounters are also characterized by an initial distance of separation R_0 much larger than the firing range d.

The first phase of the solution of such a differential game is to apply the set of necessary conditions for an assumed saddle-point optimality, generating a "candidate" for the solution. Then it has to be verified that the saddle-point inequality of Eq. (31), or some equivalent sufficiency condition, is indeed satisfied. For differential games of complex dynamic structure the verification phase is prohibitive; therefore, most investigations[7-14,17-19] result only in "candidate" solutions.

To generate a candidate solution from some initial conditions of the game, a 12-dimensional, nonlinear, two-point boundary-value problem has to be solved. Several numerical techniques were developed to solve this problem,[7-14] all requiring an excessive computational effort impractical for a rapid systematical assessment, as well as for real-time implementation. For such purposes an approximate analytical solution is preferred. Such an approximation was obtained by the method of forced singular perturbations detailed in several previous publications.[3-7,21,22] In the next subsection the FSP methodology and its main results are briefly summarized.

C. Zeroth-Order Feedback Approximation

The approximation is based on applying the method of singular perturbations as described in Refs. 2-7. First, a singularly perturbed dynamic model has to be defined. Analyzing this model to zeroth-order leads to a set of lower-dimensional problems that can be solved analytically. Based on these analytical subgame solutions, a uniformly valid zeroth-order approximation can be synthesized in a feedback form for the entire game.

The success of the singular-perturbation approach depends on the time-scale separation of the state variables. In this respect, inspection of the set of differential equations (1) and (2) and (7-10) leads to the following observations.

1) If the separation distance R is sufficiently large, as assumed for a medium-range interception, the turning rate of the line of sight, given by Eq. (2), will be very slow compared with the turning rates of the airplanes given in Eqs. (9) and (10).

2) Longitudinal accelerations of an airplane in Eqs. (7) and (8) are much smaller (on the order of ± 0.5 g or less) than the lateral accelerations (limited to 6-8 g) used for turning in Eqs. (9) and (10).

3) Equations (1) and (2), describing the relative geometry in polar coordinates, are strongly connected and have to be analyzed on the same time scale.

Based on these observations the following hierarchy of state variables can be established: R, ψ (slowest); M_P, M_E (faster); and χ_P, χ_E (fastest), leading to a singularly perturbed dynamic model. Such a model can be obtained either by proper

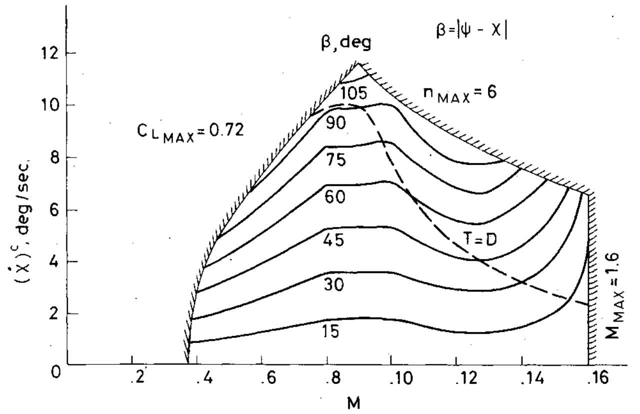

Fig. 2 Feedback chart of zeroth-order FSP turning strategy.

rescaling of the variables, as in Ref. 2, or by artificial insertion of the perturbation parameter ϵ, as in Refs. 3 and 4. It has been demonstrated that the second approach, also called a "forced" singular-perturbation technique, leads to a zeroth-order solution which is identical to the result of a scaling transformation. Because FSP seems to be much more convenient for multiple time-scale problems, it is the approach adopted for investigation of the horizontal interception game. Such an analysis is presented in detail in previous papers.[3,21,22] Here only the essential steps and the main results are summarized.

The zeroth-order FSP analysis of the present differential game leads to a sequence of three lower-order subgames:

1) Reduced-order game where each airplane can control its speed and direction instantaneously.

2) Velocity boundary-layer game assuming instantaneous direction control in a frozen relative geometry.

3) Turning boundary-layer game assuming constant speed and geometry.

Each subgame yields a simple analytical solution. Moreover, each boundary-layer game can be decoupled into two independent optimal control problems of the participating aircraft.

The solution of the reduced-order game is almost a trivial one: a "tail chase" flown by each airplane orienting its velocity vector to the line of sight at maximum speed. In order to reach these flight conditions the velocity boundary layer requires the use of full thrust.

$$\xi_P = \xi_E = 1 \tag{34}$$

The turning boundary layer also yields a closed-form solution relating the required turning rate $(\dot{\chi})^c$ of each airplane to its sustained $(T=D)$ turning performance $(\dot{\chi})_s$ as an *explicit* function of the state variables:

$$(\dot{\chi})^c = (\dot{\chi})_s \left(\frac{2M}{M^r - M}\right)^{\frac{1}{2}} \sin\left(\frac{\xi - \chi}{2}\right) \tag{35}$$

In this last feedback formula derived for the initial conditions but which is uniformly valid for the whole interception, M^r represents the Mach number of the respective airplane in the reduced-order game which is M_{max}. Such a feedback control strategy (not requiring even range measurements) is depicted in Fig. 2 for constant values of $\beta = |\psi - \chi|$. In the same figure the sustained turning rate $(\dot{\chi})_s$ is shown by the dashed curve.

Implementation of the above zeroth-order control strategies for any set of initial conditions generates a pair of suboptimal trajectories, and finally (when capture takes place) an approximate value of the cost function is obtained. The accuracy of this approximation is assessed by direct comparison to the exact optimal solution obtained in an open-loop form[17-19] by backward integration.

D. Description of the Open-Loop Optimal Solution

The original horizontal interception game formulated in this section had six dimensions. It is well known that since aircraft dynamics are independent of the horizontal flight direction, the order of the system can be reduced to five by using relative angular coordinates. In Refs. 17-19 directions are measured relative to the line of sight, and the final line-of-sight direction is used as reference. In this particular coordinate system the necessary conditions of game optimality are decoupled into two one-sided optimal control problems that can be solved independently. This approach enables one to generate extremal trajectories for each airplane, independently of its role, the position of its opponent and capture conditions. Such an open-loop trajectory is obtained by simultaneous retrograde integration of the state and adjoint equations from a given set of end conditions. In a game solution, two different extremal trajectories can be superposed. They have to be separated by the distance of capture measured along the common final line of sight, assuming that the capture condition expressed by Eq. (28) is satisfied. The construction of an individual extremal trajectory is essentially identical to the method proposed and explored for the solution of a time-optimal turn to a point with an unspecified final direction.[23]

In order to obtain the extremal trajectory corresponding to any given set of initial conditions, a systematic search has to be carried out in the parameter space of the final conditions (final speed, direction, and time). Because matching a given set of initial conditions may require an excessive computational effort, only *existing* extremal trajectories were used for comparison. Any point along such an extremal can be considered as an initial point of an optimal trajectory.

The existing open-loop extremals had been calculated using "normalized" distances (divided by the speed of sound in order to be consistent with the representation of velocities by Mach numbers) in a Cartesian coordinate frame.

For the sake of convenience in making the comparison, the digital simulation, which implemented the zeroth-order FSP feedback control strategies, used the same normalized variables.

The aircraft model used in the simulation and the criteria for selecting examples for comparison are discussed in the next section.

III. Aircraft Model and Conditions of Comparison

In order to comply with the limitation of using only existing results, all computations were performed for an altitude of 20,000 ft, implying $a = 316.1$ m/s and $P = 4751.5$ kg/m^2. For the same reason, an identical aircraft model (an early version of the F4-B, used in Ref. 23) served both for the pursuer and the evader. General aircraft information is summarized in Table 1, and the aerodynamic and propulsion data at $h =$ 20,000 ft are given in Table 2 for Mach numbers in the range 0.6-1.6.

Since the singular-perturbation methods assume a time-scale separation between trajectory and airplane dynamics, only open-loop extremals of relatively long duration could be considered as candidates for meaningful comparison. As a reasonable limit, $t_f^* \geq 50$ s was selected.

The terminal and initial conditions of the available extremal trajectories are summarized in Table 3. For all cases, the optimal throttle setting is maximal ($\xi^* = 1$), consistent with the FSP solution described in Sec. II.C. From these five extremals, four trajectory pairs which satisfy Eq. (28) were mated to constitute optimal game solutions. Respective points on both trajectories with the same retrotime $(t_f - t)$ were selected as initial conditions of the pursuit-evasion game.

The process of matching the relative geometry required a translation of the pursuer's trajectory

$$\hat{X}_P' = \hat{X}_P + \Delta\hat{X} \tag{36}$$

$$\hat{Y}_P' = \hat{Y}_P + \Delta\hat{Y} \tag{37}$$

in order to satisfy the lateral matching condition

$$\hat{Y}_E(t_f^*) - \hat{Y}_P'(t_f^*) \triangleq \hat{Y}(t_f^*) = 0 \tag{38}$$

and, using Eq. (27), the longitudinal condition,

$$\hat{X}_E(t_f^*) - \hat{X}_P'(t_f^*) \triangleq \hat{X}(t_f^*) = \hat{R}_f = \hat{d} \tag{39}$$

Consequently, the initial conditions of the game with matched relative geometry were obtained by

$$\hat{X}_0 = -\Delta\hat{X} = \hat{X}_P(t_f^*) - \hat{X}_E(t_f^*) + \hat{d} \tag{40}$$

$$\hat{Y}_0 = -\Delta\hat{Y} = \hat{Y}_P(t_f^*) - \hat{Y}_E(t_f^*) \tag{41}$$

For the selected initial condition pairs listed in Table 4, the best turning radii of the airplanes were computed. For any pair of matched extremals, the constant value of $(\hat{X}_0 - \hat{d})$, computed from Eq. (40), is also listed. By varying the "normalized capture range" $\hat{d}$, a large set of differential games with different initial ranges was formulated.

For the constant-speed game model[2] it was shown that the small parameter, characterizing the time-scale separation between trajectory and aircraft dynamics, is the ratio of the minimum turning radius r of the airplane to the initial distance of separation

$$\epsilon_g \triangleq r/R_0 \tag{42}$$

Although, in the variable-speed case, the technique of "forced" singular perturbations (an artificial insertion of a perturbation parameter $\epsilon = 1$) was used, such a ratio still provides a measure of the time-scale separation. The validity of the singular-perturbation approach depends on the value of this parameter. For each pair of trajectories the capture ranges were selected in such a way that $0.1 < \epsilon_g < 0.25$. For sake of uniformity the turning radius of the pursuer was used as a reference.

IV. Comparison of Results

A. Qualitative Comparison

Analysis of the open-loop extremals allows certain qualitative conclusions to be drawn. The very nature of the independent extremals exhibits a strong similarity to the zeroth-order FSP solution. As was pointed out in Sec. III, the forced singular-perturbation technique leads to a decoupling of the original differential game to a "simple pursuit" game and two independent sets of multiple time-scale control problems optimizing an accelerating turn to the instantaneous line of sight. The exact open-loop optimal solution does the same, but relative to the final direction of the line of sight. If a time-scale separation between airplane turning dynamics and variations of the relative geometry is a valid assumption, then the current direction of the line of sight presents a good approximation to its final direction. In such a case, it can be expected that the accuracy of the zeroth-order feedback approximation would be satisfactory.

The control strategies obtained by the two different methods are indeed similar. Both require full thrust for time-optimal interception until the maximum speed limit is attained. (In the examples used, the speed limits were never reached.) The turning strategies are characterized by a gradually decreasing rate of turn as the flight direction asymptotically reaches the required value. The difference between the exact and the zeroth-order solution is the reference direction. The larger the difference between the instantaneous and final directions of the line of sight, the worse the accuracy of the FSP approximation. Development of first-order FSP corrections would improve the zeroth-order solution. Feedback implementation of such a method now seems feasible.[24]

A second source of inaccuracy can be expected to exist from the inspection of Eq. (35) and Tables 1 and 3. In all of the extremal trajectories available for comparison (see Table 3) the final Mach number M_f is not equal to M_{max} as predicted by the zeroth-order FSP solution. This discrepancy indicates that the velocity dynamics are not "fast" enough compared to the variations of the relative geometry. The insufficient time-scale separation expressed by large differences of $(M_{max} - M_f)$ can generate errors in the turning strategy given by Eq. (35). Consequently, the outcome of the interception game will be inaccurate.

Table 1 General aircraft information

Weight, W	15,875 kg (25,000 lb)
Wing area, S	49.23 m² (530 ft²)
Wing loading, (W/S)	322.4 kg/m² (66 psf)
Maximum Mach No. (at 20,000 ft), M_{max}	1.6
Maximum load factor, n_{max}	6.0
Maximum usable lift coefficient, $C_{L_{max}}$	0.72

Table 2 Aerodynamic and propulsion data

	Mach No.					
Parameter	0.6	0.8	1.0	1.2	1.4	1.6
C_{D_0}	0.013	0.013	0.014	0.041	0.039	0.036
K	0.157	0.157	0.180	0.247	0.296	0.343
T_{max}/W [a]	0.491	0.566	0.666	0.780	0.903	1.02

[a]Altitude = 20,000 ft; speed of sound = 316.1 m/s; static pressure = 4751.5 kg/m².

Table 3 End conditions of extremal trajectories ($\hat{X}_0 = \hat{Y}_0 = 0$)

Trajectory No.	M_f	M_0	t_f^*, s	χ_0, rad	χ_f, rad	$\hat{X}_f$ [a]	$\hat{Y}_f$ [a]
I	1.5	1.238	97.1	1.330	1.7×10^{-7}	126.68	13.21
II	1.4	1.174	97.1	3.077	1.7×10^{-7}	97.30	16.48
III	1.377	1.2	50.0	9.550	1.2×10^{-5}	63.78	4.83
IV	1.272	1.2	50.0	2.280	3.5×10^{-5}	47.11	14.20
V	1.222	0.9	60.0	2.917	5.2×10^{-6}	49.05	11.31

[a]One unit is 316.1 m.

Table 4 Selected initial conditions for comparison

Example No.	Trajectory No. P	Trajectory No. E	M_{P_0}	χ_{P_0}, rad	$\hat{r}_P$ [a]	M_{E_0}	χ_{E_0}, rad	$\hat{r}_E$ [a]	$(\hat{X}_0 - \hat{d})$ [a]	t_f^*, s
1	I	II	1.238	1.33	8.35	1.174	3.077	7.51	29.38	97.1
2	III	IV	1.20	0.55	7.84	1.20	2.280	7.84	16.67	50.0
3	III	V	1.20	0.55	7.84	0.822	0.960	4.43	21.23	50.0
4	II	V	1.167	0.03	7.42	0.90	2.917	4.41	28.62	60.0

[a]One unit is 316.1 m.

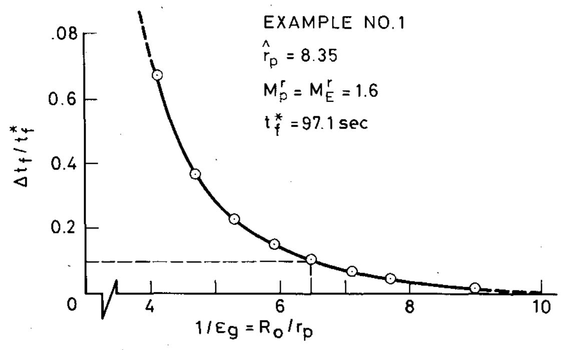

Fig. 3 Payoff accuracy of the zeroth-order FSP solution: example 1.

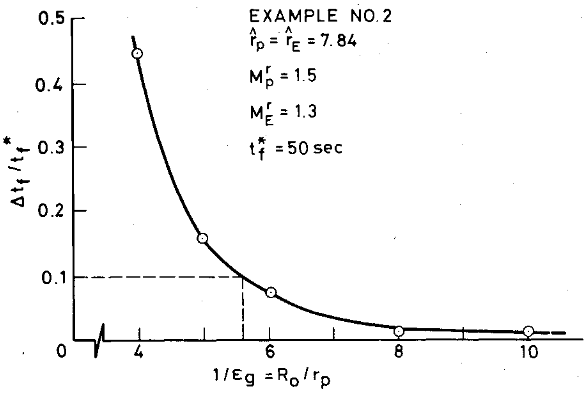

Fig. 4 Payoff accuracy of the zeroth-order FSP solution: example 2.

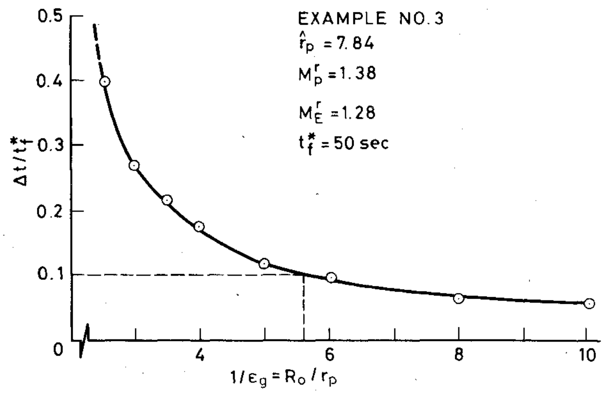

Fig. 5 Payoff accuracy of the zeroth-order FSP solution: example 3.

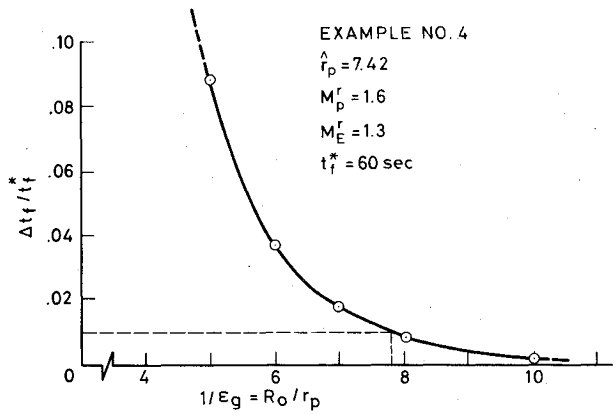

Fig. 6 Payoff accuracy of the zeroth-order FSP solution: example 4.

Such errors can be minimized, however, by making a rough estimate of the actual final speeds of the engagement. In an on-line implementation scheme this can be done by simple geometric considerations as a function of the initial conditions of the game. The result of this quickly obtained estimate then has to be inserted into the feedback formula of Eq. (35) as the value of M^r.

B. Quantitative Comparison of Payoff Accuracy

The purpose of such quantitative comparison is to determine the domain of initial conditions for which the accuracy of the zeroth-order FSP feedback solution is satisfactory. If this domain contains a region of practical interest, then the approximation can be considered useful.

The examples defined in Table 4 were recomputed with different initial and terminal ranges, keeping the relative "distance to close" constant. In Fig. 3 the relative payoff accuracy of the zeroth-order FSP feedback approximation, defined as,

$$\frac{\Delta t_f}{t_f^*} \triangleq \frac{t_f - t_f^*}{t_f^*} \tag{43}$$

is plotted for the first example vs the inverse of the geometric perturbation parameter of the pursuer

$$\frac{1}{(\epsilon_g)_P} \triangleq \frac{R_0}{r_P(M_{P_0})} \tag{44}$$

the ratio of the initial range divided by the pursuer's best turning radius at its initial Mach number, M_{P_0}. In this example the zeroth-order assumption of $M^r = M_{max}$ was kept unchanged.

In Figs. 4-6 similar results for three additional examples of Table 4 are plotted. These examples are characterized by the fact that the final Mach number of the extremal trajectory is very different from the value of $M_{max} = 1.6$. Such a discrepancy can result in control strategies that deviate strongly from the optimal strategy. In these examples this difficulty was overcome by substituting a rough estimate of M_f for the value of M^r in the respective control strategies. The values of the adjusted parameters M_P^r and M_E^r are noted in the respective figures. All examples show that an accuracy of 1% is obtained if ϵ_g is on the order of one-eighth or less. Such a level of precision seems to be highly satisfactory for all practical purposes (aircraft models are generally known with much lower accuracy).

In all of the available examples the pursuer has less of a turning requirement and reaches the vicinity of the line of sight earlier than the evader. In the subsequent phase the line-of-sight rotation is induced by the evader's maneuver. The suboptimal FSP strategy of the pursuer requires following of this rotation, which can be in the direction opposite to the initial maneuver. Such a control strategy is a substantial deviation from the optimal solution, which requires a turn toward the *final* line of sight. At the other end the evader's turning strategy in this case is similar, at least qualitatively, to the optimal strategy. Consequently, the capture time obtained in these examples by the FSP strategy pair is always longer than the optimal time.

V. Conclusions

Results of the comparison, presented and discussed in this paper, lead to the conclusion that the method of forced singular perturbations provides a valid mathematical tool for analyzing medium- and longer-range pursuit-evasion engagements. The demonstrated accuracy of the zeroth-order forced singular-perturbation approximation is very encouraging.

The turning radius of an airplane at speeds higher than the corner velocity (for a 6-*g* limit load factor and an altitude of 20,000 ft) is approximately 1.5 km for sonic flight and is proportional to the square of the velocity. The maximum firing range of advanced air-to-air missiles in a tail-chase situation (on the order of 6-8 km at such an altitude) is typically larger than the aircraft turning radius. As a consequence, side-stepping maneuvers which would have invalidated the presented approach are not optimal.

Results of the reported comparison show that a payoff accuracy of 1% is guaranteed if the value of the geometric perturbation parameter is less than 0.125 and the final velocities are at least roughly estimated. Consequently, the zeroth-order forced singular-perturbation analysis is compatible with medium-range interceptions in a real operational air-to-air scenario.

It has to be admitted, however, that the validation of the forced singular-perturbation approximation presented in this report is by no means comprehensive. Because of inherent limitations, only horizontal engagements were analyzed, only a single airplane model was investigated, and the set of initial conditions was limited. Moreover, the eventual necessity of final speed estimation indicates that further improvements may be required.

Nevertheless, the attractiveness of the forced singular-perturbation approach is emphasized by the fact that it provides a reasonably accurate explicit feedback control strategy for onboard applications. The explicit form enhances the insight; and, because of the simplicity of the implementation, it can be incorporated in any future integrated fire and flight control system with virtually no additional effort or cost.

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